EXISTENCE OF SOLITARY WAVES IN A NONLINEAR THERMOELASTIC MEDIUM WITH PRIOR DEFORMATIONS

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In a geometrically one-dimensional nonlinear formulation, the existence of solitary waves of deformations in nonlinear thermoelastic previously stressed medium is studied. The speed of propagation of the waves is shown to depend on the initial deformations.

1. Basic Equations of Nonlinear Thermoelasticity. Let a body be subject to the laws of nonlinear thermoelasticity theory [1-5], then studying wave processes in one-dimensional formulation in Lagrangian variables reduces to solving the following equations [5]:

the equation of motion

$$\frac{\partial}{\partial x}[(1+\varepsilon)\sigma^*] = \rho_0 \frac{\partial^2 u}{\partial t^2},$$

or

$$\frac{\partial^2}{\partial x^2}[(1+\varepsilon)\sigma^*] = \rho_0 \frac{\partial^2 \varepsilon}{\partial t^2}, \tag{1}$$

where $\varepsilon = \partial u/\partial x$;

the equation of thermal conductivity

$$\frac{\partial s}{\partial t} = k^* \frac{\partial}{\partial x}\left[\frac{\partial \theta}{\partial x}\frac{1}{(1+\varepsilon)}\right]. \tag{2}$$

Here $k^* = k/\rho_0 T_0$, k is a constant, $\theta = T - T_0$, $|T - T_0|/T_0 \ll 1$.

the relation of deformation and transport:

$$e = \varepsilon + \frac{1}{2}\varepsilon^2. \tag{3}$$

the defining relations:

$$\sigma^* = \frac{\partial F}{\partial e},\ S = -\frac{\partial F}{\partial \theta},\ F = F(e,\ \theta), \tag{4}$$

where F is the free energy.

2. Basic Equations of the Nonlinear Theory of Thermoelasticity with Prior Deformations. Let $\bar{\sigma}^*$, $\bar{e}$, $\bar{\varepsilon}$, $\bar{S}$, $\bar{\theta}$ be characteristics of the stress–strain state of a body at a particular point in time, and σ_0^*, e_0, ε_0, S_0, θ_0 be the same quantities in the initial state, and σ^*, e, ε, S, θ their perturbations. Then

$$\bar{\sigma}^* = \sigma_0^* + \sigma^*,\ \bar{e} = e_0 + e,\ \bar{\varepsilon} = \varepsilon_0 + \varepsilon,\ \bar{S} = S_0 + S,\ \bar{\theta} = \theta_0 + \theta. \tag{5}$$

Let us assume:

$$S_0 = \text{const},\ \theta_0 = 0,\ \varepsilon_0 = \text{const},\ \sigma_0^* = \text{const}. \tag{6}$$

Using (5) and (6), Eqs. (1)-(4) take the form

$$\frac{\partial^2}{\partial x^2}[(1+\varepsilon)\sigma^* + \varepsilon\sigma_0^*] = \rho_0 \frac{\partial^2 \varepsilon}{\partial t^2}, \tag{7}$$

$$\frac{\partial S}{\partial t} = k^* \frac{\partial}{\partial x}\left[\frac{\partial \theta}{\partial x}\frac{1}{(1+\varepsilon+\varepsilon_0)}\right], \tag{8}$$

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$$e+e_0=\varepsilon+\varepsilon_0+\frac{1}{2}(\varepsilon+\varepsilon_0)^2,\ e_0=\varepsilon_0+\frac{1}{2}\varepsilon_0^2, \tag{9}$$

$$\sigma^*+\sigma_0^*=\frac{\partial F}{\partial(e+e_0)},\ \bar{S}=-\frac{\partial F}{\partial\theta},\ F=F(e+e_0,\ \theta). \tag{10}$$

Below we shall examine the problem of choosing the function $F(e + e_0, \theta)$ for which system (7)-(10) has a solitary wave (soliton) type solution for the deformation e. Therefore, it is necessary to add the following equations, which have soliton characteristics [6], to those cited above:

$$\frac{\partial^2 e}{\partial\bar{x}^2}-[K+\tilde{u}(\bar{x},t)]\,e=0 \tag{11}$$

or

$$\frac{\partial\tilde{u}}{\partial t}-6\tilde{u}\frac{\partial\tilde{u}}{\partial\bar{x}}+\frac{\partial^3\tilde{u}}{\partial\bar{x}^3}=0, \tag{12}$$

$$\frac{\partial e}{\partial t}=-4\frac{\partial^3 e}{\partial\bar{x}^3}+6\tilde{u}\frac{\partial e}{\partial\bar{x}}+3e\frac{\partial\tilde{u}}{\partial\bar{x}}, \tag{13}$$

where K is a constant and $\bar{x} = (1 + \varepsilon_0)x$.

3. Single-Soliton Waves in a Nonlinear Prestressed Thermoelastic Medium. We shall assume that

$$e=e(\bar{x}+ct),\ \sigma^*=\sigma^*(e),\ S=S(e),\ \theta=\theta(e), \tag{14}$$

where c is the speed of the solitary waves.

With this assumption, system (11)-(13) has the following soliton solution [6]:

$$e=\beta\,cn\left\{\left[\bar{F}(a)-\frac{R\beta}{a}(\bar{x}+ct)\right],\ a\right\}. \tag{15}$$

Here

$$\beta^2=\frac{1}{2}\left(\sqrt{\Delta}+\frac{K}{R^2}\right),\ \alpha^2=\frac{1}{2}\left(\sqrt{\Delta}-\frac{K}{R^2}\right),$$
$$\Delta=\frac{K^2}{R^4}+\frac{4c_0^2}{R^2},\ c=-4K,\ a^2=\beta^2/(\alpha^2+\beta^2), \tag{16}$$

R, K, c are constants which are determined from the initial conditions, $\bar{F}(a)$ is a complete elliptic integral of the 1st kind, and cn{z, a} is a Jacobi elliptic function.

Using (14), from Eqs. (7)-(9) we obtain

$$\sigma^*+\sigma_0^*=\bar{\rho}_0c^2\left[1-\frac{1}{\sqrt{1+2(e+e_0)}}\right]+\frac{m}{\sqrt{1+2(e+e_0)}}\int_0^{e+e_0}\frac{d\tau}{\tilde{g}(\tau)}, \tag{17}$$

$$\sqrt{1+2e_0}\,\sigma_0^*=\bar{\rho}_0c^2(\sqrt{1+2e_0}-1)+m\int_0^{e_0}\frac{d\tau}{g(\tau)}, \tag{18}$$

$$S=\frac{\bar{k}g(e)}{c\sqrt{1+2(e+e_0)}}\frac{d\theta}{de}, \tag{19}$$

where

$$\bar{\rho}_0=\rho_0(1+2e_0)^{-1};\ \bar{k}=k^*(1+2e_0); \tag{20}$$

m is an integration constant, $m \geq 0$;

$$g(\tau)=R[A(\tau)]^{\frac{1}{2}},\ A(\tau)=(\alpha^2+\tau^2)(\beta^2-\tau^2), \tag{21}$$

$$\tilde{g}(\tau)=R[B(\tau)]^{\frac{1}{2}},\ B(\tau)=-\tau^4+4e_0\tau^3+(\beta^2-\alpha^2-6e_0^2)\tau^2+$$
$$+2e_0\tau(2e_0^2+\alpha^2-\beta^2)-e_0^4+(\beta^2-\alpha^2)e_0^2+\alpha^2\beta^2. \tag{22}$$

If $0 \le e_0 \le \beta$, $0 \le e \le \beta$, then it is easy to show that

$$A(\tau) \geqslant 0 \ \forall\, \tau \in [0, e_0], \ B(\tau) \geqslant 0 \ \forall\, \tau \in [0, e+e_0]. \tag{23}$$

In deriving Eqs. (17)-(19) we used the assumptions that $\varepsilon \ge 0$, $\varepsilon_0 \ge 0$, and

$$\varepsilon + \varepsilon_0 = [1 + 2(e+\varepsilon_0)]^{\frac{1}{2}} - 1, \ \varepsilon_0 = [1 + 2e_0]^{\frac{1}{2}} - 1. \tag{24}$$

From Eq. (10) we have

$$\frac{\partial F}{\partial (e+e_0)}\bigg|_{\theta=y(e,e_0)} = \bar{\rho}_0 c^2\left[1 - \frac{1}{\sqrt{1+2(e+e_0)}}\right] + \frac{m}{\sqrt{1+2(e+e_0)}} \int\limits_0^{e+e_0} \frac{d\tau}{\tilde{g}(\tau)}, \tag{25}$$

$$-\frac{\partial F}{\partial \theta}\bigg|_{\theta=y(e,e_0)} = \frac{\bar{k} g(e)}{c\sqrt{1+2(e+e_0)}} \frac{d\theta}{de} + S_0. \tag{26}$$

Equations (25) and (26) determine the conditions of the existence of the function $F(e + e_0, \theta)$, for which solitary waves propagate in a nonlinear thermoelastic medium with prior deformations.

For example, let

$$F = f_1(e+e_0) - \gamma(e+e_0)\theta - \frac{\varkappa}{2}\theta^2, \tag{27}$$

$f_1(e + e_0)$ should be determined from system (25), (26) along with the function $\theta = y(e, e_0)$ (γ, κ are constants).

From Eqs. (25)-(27) we obtain

$$f_1'(e+e_0) - \gamma y(e, e_0) = \bar{\rho}_0 c^2\left[1 - \frac{1}{\sqrt{1+2(e+e_0)}}\right] + \frac{m}{\sqrt{1+2(e+e_0)}} \int\limits_0^{e+e_0} \frac{d\tau}{\tilde{g}(\tau)}, \tag{28}$$

$$\gamma(e+e_0) + \varkappa y(e, e_0) = \frac{c^{-1}\bar{k} g(e)}{\sqrt{1+2(e+e_0)}} \frac{dy}{de}. \tag{29}$$

From Eq. (29) follows

$$\frac{dy}{de} = \Pi(e)\, y + \Gamma(e). \tag{30}$$

Here

$$\Pi(e) = \frac{\varkappa c\sqrt{1+2(e+e_0)}}{\bar{k} g(e)}, \tag{31}$$

$$\Gamma(e) = \frac{\gamma c(e+e_0)\sqrt{1+2(e+e_0)}}{\bar{k} g(e)}. \tag{32}$$

From Eq. (30) follows

$$y = \exp\left(\int\limits_0^e \Pi(\tau)\, d\tau\right) \int\limits_0^e \frac{\Gamma(\tau)\, d\tau}{\exp\left(\int\limits_0^\tau \Pi(t)\, dt\right)}. \tag{33}$$

Substituting (33) into (28), we obtain

$$f_1(e+e_0) = \bar{\rho}_0 c^2(e+e_0) - \bar{\rho}_0 c^2 \ln\sqrt{1+2(e+e_0)} + \\ + \gamma^* \int\limits_0^{e+e_0} E(\tau)\, d\tau \int\limits_{e_0}^{\tau} \frac{(t-e_0)\sqrt{1+2t}}{E(t)\,\tilde{g}(t)}\, dt + m \int\limits_0^{e+e_0} \frac{d\tau}{\sqrt{1+2\tau}} \int\limits_0^{\tau} \frac{dt}{\tilde{g}(t)}. \tag{34}$$

Here

$$E(\tau) = \exp\left(\varkappa^* \int\limits_{e_0}^{\tau} \frac{dt}{\tilde{g}(t)}\right), \tag{35}$$

$$\varkappa^* = \varkappa c/\bar{k}, \ \gamma^* = \gamma^2 c/\bar{k}. \tag{36}$$

Consequently, in this case the desired solution is given by the formulas

$$\sigma^* = \bar{\rho}_0 c^2 \left[1 - \frac{1}{\sqrt{1 + 2(e + e_0)}} \right] + \frac{m}{\sqrt{1 + 2(e + e_0)}} \int_0^{e+e_0} \frac{d\tau}{\tilde{g}(\tau)} - \sigma_0^*, \tag{37}$$

where

$$\sigma_0^* = \bar{\rho}_0 c^2 \left[1 - \frac{1}{\sqrt{1 + 2e_0}} \right] + \frac{m}{\sqrt{1 + 2e_0}} \int_0^{e_0} \frac{d\tau}{g(\tau)}. \tag{38}$$

Here the function $\theta = y(e, e_0)$ is determined by Eqs. (31)-(33), the entropy $S(e, e_0)$ is calculated from Eq. (19), and the free energy F is determined by Eq. (27), for which the function $f_1(e + e_0)$ is calculated from Eq. (34) and the functions $\tilde{g}(\tau)$, $g(\tau)$ are determined by Eqs. (21) and (22). The dependence of the speed of the soliton waves on initial deformations is expressed by the following formula:

$$\bar{\rho}_0 c^2 = \frac{(1 + \varepsilon_0)}{\varepsilon_0} \sigma_0^* - \frac{m}{\varepsilon_0} \int_0^{e_0} \frac{d\tau}{g(\tau)}. \tag{39}$$

In conclusion, we point out that this work is closely related to [6, 7].

NOTATION

T is the absolute temperature of the medium at a given point in time t; T_0 is its value at the initial point in time t_0; S is the entropy of the medium; x is the Lagrangian coordinate of a point in the medium; ρ_0 is the density of the materials in the medium in the undeformed state; σ^* is the component of the generalized stress tensor in the Ox direction; u is the displacement of a point in the medium along the Ox-axis; e is the deformation of the medium along the Ox-axis.

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